The role of measurement in quantum games

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Abstract
The game of prisoner dilemma is analysed to study the role of measurement basis in quantum games. Four different types of payoffs for quantum games are identified on the basis of different combinations of initial state and measurement basis. A relation among these different payoffs is established.

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1. Introduction

Game theory deals with a situation in which two or more parties compete to maximize their respective payoffs by playing suitable strategies according to the known payoff matrix. Its extension to the quantum domain made it more interesting where the quantum strategies have shown a clear advantage over classical strategies [1–3, 5]. For further reading on classical and quantum game theory see [6, 7].

In the quantum version of the game, an arbiter prepares an initial quantum state and passes it on to the players. After applying their local operators (or strategies), the players return the state to the arbiter who then announces the payoffs by performing a measurement with the application of suitable payoff operators depending on the payoff matrix of the game. The role of the initial quantum state remained an interesting issue in quantum games [2–5]. However, the importance of the payoff operators used by the arbiter to perform measurement to determine the payoffs of the players remained mostly unnoticed. In our earlier paper [8], we have pointed out the importance of measurement basis in quantum games. It was shown that if the arbiter is allowed to perform the measurement on an entangled basis, interesting situations could arise which were not possible in the framework of Eisert *et al* [2] and Marinatto and Weber [3] schemes. In this paper, we further extend our earlier work to investigate the role of measurement basis in quantum games by taking the prisoner dilemma as an example. Quantum payoffs are divided into four different categories on the basis of initial state and measurement basis. These different situations arise due to the possibility of having product or entangled initial state and then applying product or entangled basis for the measurement [9, 10]. In the

context of our generalized framework for quantum games (see [8]), the four different types of payoffs are

(i) $\$_{PP}$ is the payoff when the initial quantum state is of the product form, and product basis is used for measurement to determine the payoff.
(ii) $\$_{PE}$ is the payoff when the initial quantum state is of the product form, and entangled basis is used for measurement to determine the payoff.
(iii) $\$_{EP}$ is the payoff when the initial quantum state is entangled, and product basis is used for measurement to determine the payoff.
(iv) $\$_{EE}$ is the payoff when the initial quantum state is entangled, and entangled basis is used for measurement to determine the payoff.

Our results show that these payoffs obey a relation, $\$_{PP} < \$_{PE} = \$_{EP} < \$_{EE}$ at the Nash equilibrium (NE). It is interesting to note that this relation is very similar to an existing relation for the classical capacities of the quantum channels. It is shown in [11] that for a quantum channel the capability to transmit maximum classical information, called the classical channel capacity C of a quantum channel, a relation of the form $C_{PP} < C_{PE} = C_{EP} < C_{EE}$ holds. In this paper, we have not tried to investigate the possible relationship between channel capacity and payoffs.

2. Prisoner dilemma

In the game of prisoner dilemma, two prisoners are being interrogated in separate cells for a crime they have committed together. The two possible moves for these prisoners are to cooperate (C) or to defect (D). They are not allowed to communicate but have access to the following payoff matrix:

$$\text{Alice} \begin{array}{c} \\ C \\ D \end{array} \begin{array}{c} \overset{\text{Bob}}{} \\ \overset{C \qquad D}{\begin{bmatrix} (3,3) & (0,5) \\ (5,0) & (1,1) \end{bmatrix}} \end{array}. \tag{1}$$

It can be seen from equation (1) that D is the dominant strategy for the two players. Therefore, rational reasoning forces each player to play D causing (D, D) as the Nash equilibrium of the game with payoffs $(1, 1)$, i.e., 1 for both. The players could have got higher payoffs had both of them decided to play C instead of D. This is the dilemma in this game [12]. Eisert *et al* [2] analysed this game in the quantum domain and showed that there exists a suitable quantum strategy for which the dilemma is resolved. They also pointed out a quantum strategy which always wins over all classical strategies.

In our generalized version of quantum games, the arbiter prepares the initial state of the form [8],

$$|\psi_{\text{in}}\rangle = \cos\frac{\gamma}{2}|CC\rangle + i\sin\frac{\gamma}{2}|DD\rangle. \tag{2}$$

Here $|C\rangle$ and $|D\rangle$ represent vectors in the strategy space corresponding to cooperate and defect, respectively, with $\gamma \in [0, \pi]$. The strategy of each of the players can be represented by the unitary operator U_i of the form

$$U_i = \cos\frac{\theta_i}{2}R_i + \sin\frac{\theta_i}{2}P_i, \tag{3}$$

where $i = 1$ or 2 and R_i, P_i are the unitary operators:

$$\begin{array}{ll} R_i|C\rangle = e^{i\phi_i}|C\rangle, & R_i|D\rangle = e^{-i\phi_i}|D\rangle, \\ P_i|C\rangle = -|D\rangle, & P_i|D\rangle = |C\rangle. \end{array} \tag{4}$$

Here we restrict our treatment to a two parameter set of strategies (θ_i, ϕ_i) for mathematical simplicity in accordance with [2]. After application of the strategies, the initial state given by equation (2) transforms to

$$|\psi_f\rangle = (U_1 \otimes U_2)|\psi_{\mathrm{in}}\rangle, \tag{5}$$

and using equations (4) and (5) the above expression becomes

$$\begin{aligned}
|\psi_f\rangle = \cos(\gamma/2)[&\cos(\theta_1/2)\cos(\theta_2/2)\,\mathrm{e}^{i(\phi_1+\phi_2)}|CC\rangle - \cos(\theta_1/2)\sin(\theta_2/2)\,\mathrm{e}^{i\phi_1}|CD\rangle \\
&- \cos(\theta_2/2)\sin(\theta_1/2)\,\mathrm{e}^{i\phi_2}|DC\rangle + \sin(\theta_1/2)\sin(\theta_2/2)|DD\rangle] + i\sin(\gamma/2) \\
\times\,[&\cos(\theta_1/2)\cos(\theta_2/2)\,\mathrm{e}^{-i(\phi_1+\phi_2)}|DD\rangle + \cos(\theta_1/2)\sin(\theta_2/2)\,\mathrm{e}^{-i\phi_1}|DC\rangle \\
&+ \cos(\theta_2/2)\sin(\theta_1/2)\,\mathrm{e}^{-i\phi_2}|CD\rangle + \sin(\theta_1/2)\sin(\theta_2/2)|CC\rangle].
\end{aligned} \tag{6}$$

The operators used by the arbiter to determine the payoff for Alice and Bob are

$$P_A = 3P_{CC} + P_{DD} + 5P_{DC}, \qquad P_B = 3P_{CC} + P_{DD} + 5P_{CD}, \tag{7}$$

where

$$\begin{aligned}
P_{CC} &= |\psi_{CC}\rangle\langle\psi_{CC}|, & |\psi_{CC}\rangle &= \cos(\delta/2)|CC\rangle + i\sin(\delta/2)|DD\rangle, & (8a) \\
P_{DD} &= |\psi_{DD}\rangle\langle\psi_{DD}|, & |\psi_{DD}\rangle &= \cos(\delta/2)|DD\rangle + i\sin(\delta/2)|CC\rangle, & (8b) \\
P_{DC} &= |\psi_{DC}\rangle\langle\psi_{DC}|, & |\psi_{DC}\rangle &= \cos(\delta/2)|DC\rangle - i\sin(\delta/2)|CD\rangle, & (8c) \\
P_{CD} &= |\psi_{CD}\rangle\langle\psi_{CD}|, & |\psi_{CD}\rangle &= \cos(\delta/2)|CD\rangle - i\sin(\delta/2)|DC\rangle, & (8d)
\end{aligned}$$

with $\delta \in [0, \pi]$. Above payoff operators reduce to that of Eisert's scheme for δ equal to γ, which represents the entanglement of the initial state [2]. And for $\delta = 0$, above operators transform into that of Marinatto and Weber's scheme [3]. Corresponding payoffs for the players, A and B, are

$$\$^A(\theta_1, \phi_1, \theta_2, \phi_2) = \mathrm{Tr}(P_A \rho_f), \qquad \$^B(\theta_1, \phi_1, \theta_2, \phi_2) = \mathrm{Tr}(P_B \rho_f), \tag{9}$$

where $\rho_f = |\psi_f\rangle\langle\psi_f|$ is the density matrix for the quantum state given by (6) and Tr represents the trace of a matrix. Using equations (6), (8) and (9), we get the following payoffs:

$$\begin{aligned}
\$^A(\theta_i, \phi_j) = {}& \sin^2(\theta_1/2)\sin^2(\theta_2/2)\left[\cos^2\left(\frac{\gamma+\delta}{2}\right) + 3\sin^2\left(\frac{\gamma-\delta}{2}\right)\right] \\
&+ \cos^2(\theta_1/2)\cos^2(\theta_2/2)[2 + \cos\gamma\cos\delta + 2\cos(2\delta(\phi_1+\phi_2))\sin\gamma\sin\delta] \\
&- \sin\theta_1\sin\theta_2\sin(\phi_1+\phi_2)[\sin\gamma - \sin\delta] + \frac{5}{4}[1 - \cos\theta_1\cos\theta_2] \\
&+ \frac{5}{4}(\cos\theta_2 - \cos\theta_1)[\cos\gamma\cos\delta + \cos(2\phi_1)\sin\gamma\sin\delta].
\end{aligned} \tag{10}$$

The payoff of player B can be found by interchanging $\theta_1 \longleftrightarrow \theta_2$ and $\phi_1 \longleftrightarrow \phi_2$ in equation (10). There can be four types of payoffs for each player for different combinations of δ and γ. In the following, $\$_{PP}(\theta_1, \theta_2)$ means payoffs of the players when the initial state of the game is product state and payoff operator used by the arbiter for measurement is also in the product form ($\gamma = 0, \delta = 0$), and $\$_{EP}(\theta_1, \theta_2, \phi_1, \phi_2)$ means the payoffs for entangled input state when the payoff operator used for measurement is in the product form, i.e., ($\gamma \neq 0, \delta = 0$). Similarly, $\$_{PE}(\theta_1, \theta_2, \phi_1, \phi_2)$ and $\$_{EE}(\theta_1, \theta_2, \phi_1, \phi_2)$ can also be interpreted. Therefore, for different values of δ and γ, the following four cases can be identified:

Case (a). When $\delta = \gamma = 0$, equation (10), becomes

$$\$_{PP}^A(\theta_1, \theta_2) = 3\cos^2(\theta_1/2)\cos^2(\theta_2/2) + \sin^2(\theta_1/2)\sin^2(\theta_2/2) + 5\sin^2(\theta_1/2)\cos^2(\theta_2/2). \tag{11}$$

This situation corresponds to the classical game where each player plays, C, with probability $\cos^2(\theta_i/2)$ with $i = 1, 2$ [13]. The Nash equilibrium corresponds to $\theta_1 = \theta_2 = \pi$, i.e., (D, D) with payoffs for both the players as

$$\$^A_{PP}(\theta_1 = \pi, \theta_2 = \pi) = \$^B_{PP}(\theta_1 = \pi, \theta_2 = \pi) = 1. \tag{12}$$

Case (b). When $\gamma = 0, \delta \neq 0$, in equation (10), then the game has two Nash equilibria one at $\theta_1 = \theta_2 = 0$ when $\sin^2(\delta/2) \geqslant \frac{2}{3}$ and the other at $\theta_1 = \theta_2 = \pi$ when $\sin^2(\delta/2) \leqslant \frac{1}{3}$. The corresponding payoffs for these Nash equilibria are

$$\begin{aligned}
\$^A_{PE}(\theta_1 = 0, \theta_2 = 0) &= \$^B_{PE}(\theta_1 = 0, \theta_2 = 0) = 3 - 2\sin^2(\delta/2), \\
\$^A_{PE}(\theta_1 = \pi, \theta_2 = \pi) &= \$^B_{PE}(\theta_1 = \pi, \theta_2 = \pi) = 1 + 2\sin^2(\delta/2).
\end{aligned} \tag{13}$$

Here in this case at NE, the payoffs are independent of ϕ_1, ϕ_2. It is clear that the above payoffs for all the allowed values of δ remain less than 3, which is the optimal payoff for the two players if they cooperate.

Case (c). For $\gamma \neq 0$, and $\delta = 0$, equation (10) again gives two Nash equilibria one at $\theta_1 = \theta_2 = 0$ when $\sin^2(\gamma/2) \geqslant \frac{2}{3}$ and the other at $\theta_1 = \theta_2 = \pi$ when $\sin^2(\gamma/2) \leqslant \frac{1}{3}$. The corresponding payoffs are

$$\begin{aligned}
\$^A_{EP}(\theta_1 = 0, \theta_2 = 0) &= \$^B_{EP}(\theta_1 = 0, \theta_2 = 0) = 3 - 2\sin^2(\gamma/2), \\
\$^A_{EP}(\theta_1 = \pi, \theta_2 = \pi) &= \$^B_{EP}(\theta_1 = \pi, \theta_2 = \pi) = 1 + 2\sin^2(\gamma/2).
\end{aligned} \tag{14}$$

It can be seen that the payoffs at both Nash equilibrium for allowed values of $\sin^2\frac{\gamma}{2}$ remain less than 3. From equations (13) and (14), it is also clear that $\$^A_{EP}(0, 0) = \$^A_{PE}(\pi, \pi)$ only for $\delta = \gamma$.

Case (d). When $\gamma = \delta = \pi/2$, equation (10) becomes

$$\begin{aligned}
\$^A_{EE}(\theta_1, \theta_2, \phi_1, \phi_2) &= 3[\cos(\theta_1/2)\cos(\theta_2/2)\cos(\phi_1 + \phi_2)]^2 \\
&\quad + [\sin(\theta_1/2)\sin(\theta_2/2) + \cos(\theta_1/2)\cos(\theta_2/2)\sin(\phi_1 + \phi_2)]^2 \\
&\quad + 5\left[\sin(\theta_1/2)\cos\frac{\theta_2}{2}\cos\phi_2 - \cos(\theta_1/2)\sin(\theta_2/2)\sin\phi_1\right]^2.
\end{aligned} \tag{15}$$

This payoff is same as found by Eisert *et al* [2], and $\theta_1 = \theta_2 = 0, \phi_1 = \phi_2 = \frac{\pi}{2}$ is the Nash equilibrium [2] of the game that gives the payoffs for both players as

$$\$^A_{EE}\left(0, 0, \frac{\pi}{2}, \frac{\pi}{2}\right) = \$^B_{EE}\left(0, 0, \frac{\pi}{2}, \frac{\pi}{2}\right) = 3. \tag{16}$$

Comparing equations (12)–(14) and (16), it is evident that

$$\$^l_{EE}\left(0, 0, \frac{\pi}{2}, \frac{\pi}{2}\right) > \left(\$^l_{PE}(\theta_1 = k, \theta_2 = k), \$^l_{EP}(\theta_1 = k, \theta_2 = k)\right) > \$^l_{PP}(\theta_1 = \pi, \theta_2 = \pi) \tag{17}$$

and

$$\$^l_{PE}(\theta_1 = k, \theta_2 = k) = \$^l_{EP}(\theta_1 = k, \theta_2 = k) \qquad \text{for} \quad \gamma = \delta \tag{18}$$

with $k = 0, \pi$ and $l = A, B$. This expression shows the crucial role of entanglement in quantum games as it can be seen that the combination of initial entangled state with entangled payoff operators gives higher payoffs as compared to all other combinations of γ and δ.

3. Conclusion

In quantum games, the arbiter (the referee) prepares an initial quantum state and passes it on to the players. After applying their local operators (their strategies) the players return their state to the arbiter. The arbiter then performs a measurement on the final state by applying the payoff operators to determine the payoffs of the player on the basis of the payoff matrix of the game. In our earlier paper [8], we pointed out the importance of measurement in the quantum games. Here we extended our earlier work, by taking the Prisoner dilemma game as an example and showed that depending on the initial states and type of measurement (product or entangled), quantum payoffs in games can be categorized into four different types. These four categories are $\$_{PP}$, $\$_{PE}$, $\$_{EP}$ and $\$_{EE}$, where P and E are abbreviations for the product and entanglement at input and output, respectively. It is shown that there exists a relation of the form $\$_{PP} < \$_{PE} = \$_{EP} < \$_{EE}$ among different payoffs at the NE.

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